International Journal of Electrical and Electronics Research ISSN 2348-6988 (online)
Vol. 12, Issue 4, pp: (1-5), Month: October - December 2024, Available at: www.researchpublish.com

Fractional Fourier Series Expansions of Two Types of Fractional Trigonometric Functions

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DOI: https://doi.org/10.5281/zenodo.13955269

Published Date: 19-October-2024

Abstract: **In this paper, we obtain the fractional Fourier series expansions of two types of fractional trigonometric functions. A new multiplication of fractional analytic functions plays an important role in this article. In fact, our results are generalizations of ordinary calculus results.**

Keywords: **fractional Fourier series expansions, fractional trigonometric functions, new multiplication, fractional analytic functions.**

I. INTRODUCTION

Fractional calculus is a mathematical analysis tool used to study arbitrary order derivatives and integrals. It unifies and extends the concepts of integer order derivatives and integrals [1-5]. Generally, many scientists do not know these fractional integrals and derivatives, and they have not been used in pure mathematical context until recent years. However, in the past few decades, the fractional integrals and derivatives have frequently appeared in many scientific fields such as fluid mechanics, viscoelasticity, physics, image processing, economics and engineering [6-13].

The definition of fractional derivative is not unique. The commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [14-19]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with ordinary calculus.

In this paper, we obtain the fractional Fourier series expansions of the following two types of fractional trigonometric functions:

$$\{r^m cos_\alpha(mx^\alpha) + r^{m+n} cos_\alpha[(m-n)x^\alpha]\} \otimes_\alpha [1 + 2r^n cos_\alpha(nx^\alpha) + r^{2n}]^{\otimes_\alpha(-1)},$$

and

$$\{r^m sin_\alpha(mx^\alpha) + r^{m+n} sin_\alpha[(m-n)x^\alpha]\} \otimes_\alpha [1 + 2r^n cos_\alpha(nx^\alpha) + r^{2n}]^{\otimes_\alpha(-1)},$$

where $0 < \alpha \le 1, |r| < 1$, and m, n are positive integers. A new multiplication of fractional analytic functions plays an important role in this paper. In fact, our results are generalizations of classical calculus results.

II. PRELIMINARIES

At first, the definition of fractional analytic function is introduced.

Definition 2.1 ([20])**:** If x, x_0, and a_k are real numbers for all k, $x_0 \in (a, b)$, and $0 < \alpha \le 1$. If the function $f_\alpha: [a, b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} (x - x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a, b] \to R$ is continuous on closed interval $[a, b]$ and it is α-fractional analytic at every point in open interval (a, b), then f_α is called an α-fractional analytic function on $[a, b]$.

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.2 ([21])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{1}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\ . \tag{2}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &=\sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{3}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{4}$$

Definition 2.3 ([22])**:** If $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}, \tag{5}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \tag{6}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha\big(g_\alpha(x^\alpha)\big)=\sum_{n=0}^{\infty}\frac{a_n}{n!}\big(g_\alpha(x^\alpha)\big)^{\otimes_\alpha n}, \tag{7}$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha\big(f_\alpha(x^\alpha)\big)=\sum_{n=0}^{\infty}\frac{b_n}{n!}\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n}. \tag{8}$$

Definition 2.4 ([23])**:** Let $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n}=f_\alpha(x^\alpha)\otimes_\alpha\cdots\otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)=1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha(-1)}$.

Definition 2.5 ([24])**:** If $0<\alpha\leq 1$, and x is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha n}. \tag{9}$$

On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha 2n}, \tag{10}$$

and

$$sin_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2n+1)}. \tag{11}$$

Definition 2.6: If the complex number $z = p + iq$, where p, q are real numbers, and $i = \sqrt{-1}$. p, the real part of z, is denoted by $\text{Re}(z)$; q the imaginary part of z, is denoted by $\text{Im}(z)$.

Proposition 2.7 (fractional Euler's formula)**:** *Let* $0 < \alpha \leq 1$, *then*

$$E_\alpha(ix^\alpha) = cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha)\ . \tag{12}$$

Proposition 2.8 (fractional DeMoivre's formula)**:** *Let* $0 < \alpha \leq 1$, *and* k *be a positive integer, then*

$$[cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha)]^{\otimes_\alpha k} = cos_\alpha(kx^\alpha) + isin_\alpha(kx^\alpha). \tag{13}$$

III. MAIN RESULTS

In this section, we find the fractional Fourier series expansions of two types of fractional trigonometric functions. At first, we need a lemma.

Lemma 3.1: *If* m, n *are positive integers,* z *is a complex number and* $|z| < 1$. *Then*

$$\frac{z^m}{1+z^n} = \sum_{k=0}^{\infty}(-1)^k z^{nk+m} \tag{14}$$

Proof: Since $|z^n| = |z|^n < 1$, it follows that

$$\frac{z^m}{1+z^n}$$

$$= z^m \cdot \sum_{k=0}^{\infty}(-1)^k (z^n)^k$$

$$= \sum_{k=0}^{\infty}(-1)^k z^{nk+m}\ .$$ q.e.d.

Theorem 3.2: *Let* $0 < \alpha \leq 1, |r| < 1,$ *and* m, n *be positive integers, then*

$$\{r^m cos_\alpha(mx^\alpha) + r^{m+n} cos_\alpha[(m-n)x^\alpha]\} \otimes_\alpha [1 + 2r^n cos_\alpha(nx^\alpha) + r^{2n}]^{\otimes_\alpha(-1)}$$
$$= \sum_{k=0}^{\infty}(-1)^k r^{nk+m} cos_\alpha((nk+m)x^\alpha)\ . \tag{15}$$

And

$$\{r^m sin_\alpha(mx^\alpha) + r^{m+n} sin_\alpha[(m-n)x^\alpha]\} \otimes_\alpha [1 + 2r^n cos_\alpha(nx^\alpha) + r^{2n}]^{\otimes_\alpha(-1)}$$
$$= \sum_{k=0}^{\infty}(-1)^k r^{nk+m} sin_\alpha((nk+m)x^\alpha)\ . \tag{16}$$

Proof Let $z = rE_\alpha(ix^\alpha)$, then by Lemma 3.1

$$[rE_\alpha(ix^\alpha)]^{\otimes_\alpha m}\left[1 + [rE_\alpha(ix^\alpha)]^{\otimes_\alpha n}\right]^{\otimes_\alpha(-1)} = \sum_{k=0}^{\infty}(-1)^k [rE_\alpha(ix^\alpha)]^{\otimes_\alpha(nk+m)}. \tag{17}$$

Using fractional DeMoivre's formula yields

$$r^m E_\alpha(imx^\alpha)[1 + r^n E_\alpha(inx^\alpha)]^{\otimes_\alpha(-1)} = \sum_{k=0}^{\infty}(-1)^k r^{nk+m} E_\alpha(i(nk+m)x^\alpha). \tag{18}$$

By fractional Euler's formula, we have

$$r^m[cos_\alpha(mx^\alpha) + isin_\alpha(mx^\alpha)]\left[1 + r^n[cos_\alpha(nx^\alpha) + isin_\alpha(nx^\alpha)]\right]^{\otimes_\alpha(-1)}$$
$$= \sum_{k=0}^{\infty}(-1)^k r^{nk+m}[cos_\alpha((nk+m)x^\alpha) + isin_\alpha((nk+m)x^\alpha)]. \tag{19}$$

Therefore,

$$\text{Re}\left\{r^m[cos_\alpha(mx^\alpha) + isin_\alpha(mx^\alpha)]\left[1 + r^n[cos_\alpha(nx^\alpha) + isin_\alpha(nx^\alpha)]\right]^{\otimes_\alpha(-1)}\right\}$$
$$= \text{Re}\left\{\sum_{k=0}^{\infty}(-1)^k r^{(nk+m)}[cos_\alpha((nk+m)x^\alpha) + isin_\alpha((nk+m)x^\alpha)]\right\}. \tag{20}$$

And hence,

$$\{r^m cos_\alpha(mx^\alpha) + r^{m+n} cos_\alpha[(m-n)x^\alpha]\} \otimes_\alpha [1 + 2r^n cos_\alpha(nx^\alpha) + r^{2n}]^{\otimes_\alpha(-1)}$$

$$= \sum_{k=0}^{\infty}(-1)^k r^{nk+m} cos_\alpha((nk+m)x^\alpha) .$$

Similarly, since

$$\mathrm{Im}\left\{ r^m[cos_\alpha(mx^\alpha) + isin_\alpha(mx^\alpha)][1 + r^n[cos_\alpha(nx^\alpha) + isin_\alpha(nx^\alpha)]]^{\otimes_\alpha(-1)} \right\}$$

$$= \mathrm{Im}\{\sum_{k=0}^{\infty}(-1)^k r^{nk+m}[cos_\alpha((nk+m)x^\alpha) + isin_\alpha((nk+m)x^\alpha)]\}. \quad (21)$$

Thus,

$$\{r^m sin_\alpha(mx^\alpha) + r^{m+n} sin_\alpha[(m-n)x^\alpha]\} \otimes_\alpha [1 + 2r^n cos_\alpha(nx^\alpha) + r^{2n}]^{\otimes_\alpha(-1)}$$

$$= \sum_{k=0}^{\infty}(-1)^k r^{nk+m} sin_\alpha((nk+m)x^\alpha) .$$ q.e.d.

IV. CONCLUSION

In this paper, we find the fractional Fourier series expansions of two types of fractional trigonometric functions. A new multiplication of fractional analytic functions plays an important role in this article. In fact, our results are generalizations of traditional calculus results. In the future, we will continue to study the problems in engineering mathematics and fractional differential equations.

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